Further Maths

Complex Numbers

In Further Mathematics A-level, you will learn about parts of mathematics that are more 'abstract' than what you have come to expect in Maths so far. One example of this is working with complex numbers.

This document aims to give you a good starting point and familiarity with this topic. The links are to TLMaths, a channel of video tutorials that give you a good overview of each part of the material. You can use TLMaths to take notes and examples like you would in class.

What are Complex Numbers?

Remember when you used the quadratic formula, and "it broke" when you had a negative discriminant? (the $b^2 - 4ac$ bit)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You might have heard that if the discriminant is negative, and you need to square root a negative number to find the solution, the quadratic has 'no solutions'. This isn't true. It's better to say it has 'no **real** solutions'. Real numbers – denoted '$\mathbb{R}$ ' - are numbers you can place on a number line. This includes all integers, decimals, negatives, and so on.

But there's other numbers too!

By adding $i = \sqrt{-1}$ to our number system, we can then represent roots of quadratics that we couldn't previously do so with "real" numbers (i.e. numbers in $\mathbb{R}$).

- $i = \sqrt{-1}$
- An **imaginary** number is of the form bi where $b \in \mathbb{R}$, e.g. $i,\ 3i,\ -2i,\ i\pi$
- A **complex** number is of the form $a + bi$, where $a, b \in \mathbb{R}$, e.g. $1 + i,\ 3 - 2i$
- We say that a is the "real part" and b the "imaginary part" of the number.
- Real numbers are usually written as *x* or *y.* Complex numbers are usually written as *z*.

TLMAths: Introducing Complex Numbers: https://www.youtube.com/watch?v=l8M1Bu2lMAY

Writing Complex Numbers

We can write the square root of any negative in terms of *i,* using what we know about surds. Recall that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.

Examples:

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$$

$$\sqrt{-50} = \sqrt{50}\sqrt{-1} = 5\sqrt{2} \times i = 5\sqrt{2}\, i$$

Practice: Write the following in terms of *i.*

1) $\sqrt{-36}$
2) $\sqrt{-20}$
3) $\sqrt{-7}$

Simplifying Complex Numbers

We treat *i* just like any other algebraic term! We add and subtract real and imaginary parts separately.

Examples:

$$(3+4i)+(6-3i) = 3+6+4i-3i = 9+i$$

$$\left(7+\frac{1}{2}i\right)-(11-2i) = (7-11)+(\frac{1}{2}i - -2i) = -4+\frac{5}{2}i$$

$$2i-2(3i-4) = (--8)+(2i-6i) = 8-4i$$

Practice: Write these complex numbers in the form *a + bi.*

1) $(2+3i)+(4+i)$

2) $i-3(2-i)$

3) $\frac{10+4i}{2}$

TLMaths: Adding and subtracting Complex Numbers:
https://www.youtube.com/watch?v=7Hw7bmix29A

Multiplying Complex Numbers

Complex numbers can multiply together just like algebraic expressions and surds. Usually, this involves expanding multiple brackets.

Since $i = \sqrt{-1}, i^2 = -1$**.** So products of complex numbers don't need several of powers of *i* to be written out – we still get numbers in the form a + b*i*.

Examples:

$(2+3i)(3-2i)$	$(5-3i)^2$
$= \mathbf{6 - 4i + 9i - 6i^2}$	$= \mathbf{(5-3i)(5-3i)}$
$= \mathbf{6 - 4i + 9i + 6}$	$= \mathbf{25 - 30i + 9i^2}$
$= \mathbf{12 + 5i}$	$= \mathbf{25 - 30i - 9}$
	$= \mathbf{16 - 30i}$

Notice that since $i^2 = -1$:

$$i^3 = \boldsymbol{i \times i \times i = -1 \times i = -i}$$

$$i^4 = \boldsymbol{i \times i \times i \times i = -1 \times -1 = 1}$$

So no matter how many complex numbers are multiplied together, the powers of i are just multiples of i and -1.

Practice:

Write each of these expressions in the form $a+bi$ where $a, b \in \mathbb{R}$

a $(2+3i)(i+5)$ **b** $(7-i)(6-3i)$

c $i(8-3i)$ **d** $(9-4i)^2$

Complex Conjugates

Any complex number (a + b*i*) has a *complex conjugate*, which is (a - b*i*). For example, the complex conjugate of (3+2i) is (3-2i), and the complex conjugate of (1-3i) is (1+3i). The conjugate of z is written as z*.

Any complex number multiplied by its complex conjugate is a real number.

Example: Let z = (4 – 6i). Find zz*.

$$zz^* = (4-6i)(4+6i) = 16 + 24i - 24i - 36i^2 = 16 + 36 = 52.$$

<u>Dividing Complex Numbers</u>

We saw in 'simplifying complex numbers' that we can easily divide all the parts of a complex number by a real number.

For example: $\frac{25-15i}{5} = 5 - 3i$.

It's much harder to see how to divide by a complex number. But using what we know about complex conjugates, we can simply divide by a real number instead. This skill is very similar to 'rationalising the denominator' from GCSE – a complex number functions much like a surd, because $i = \sqrt{-1}$.

Example: Simplify $\frac{4+3i}{1-2i}$

$\frac{4+3i}{1-2i} \times \frac{1+2i}{1+2i}$ ← 1) Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$= \frac{(4+3i)(1+2i)}{(1-2i)(1+2i)}$

$= \frac{4+3i+8i-6}{1+2i-2i+4}$ ← 2) Expand and simplify

$= \frac{-2+11i}{5}$

$= -\frac{2}{5} + \frac{11}{5}i$ ← 3) Once you have a real number as the denominator, you can write the complex number in the form (a+bi).

Practice:

Simplify these fractions, giving your answers in the form $a+bi$ where $a, b \in \mathbb{R}$

a $\frac{3}{2+i}$ **b** $\frac{2i}{1-5i}$ **c** $\frac{1+7i}{3-i}$

d $\frac{i+3}{2i-1}$ **e** $\frac{6+3i}{i-\sqrt{2}}$ **f** $\frac{\sqrt{2}i-\sqrt{6}}{\sqrt{3}-i}$

You are given that $z_1 = 3i - 2$, $z_2 = 4 + i$

Calculate these expressions, fully simplifying your answers.

a $z_1 + z_2$ **b** $z_1 z_2$

c $\frac{z_1}{z_2}$ **d** $\frac{z_2}{z_1}$

Solving Quadratic Equations

Earlier, we saw that complex numbers can be used to solve equations that don't have real solutions. There are **always** two solutions to a quadratic equation. The solutions, or 'roots' might be equal to each other, or complex, but there is still always two!

A quadratic equation with two distinct real roots: $x^2 - 7x + 12 = 0$.

If we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 48}}{2} = \frac{7 \pm 1}{2} = 3, 4.$$

In this case the discriminant $(b^2 - 4ac)$ was positive, so the answers were both real.

A quadratic equation with two complex roots: $z^2 - 2z + 10 = 0$

Again, using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = (1 + 3i), (1 - 3i).$$

In this case the discriminant was negative, which led to each root having an imaginary part.

You can also solve equations like this by completing the square:

$$z^2 - 2z + 10 = 0$$

$$(z - 1)^2 - 1 + 10 = 0$$

$$(z - 1)^2 + 9 = 0$$

$$(z - 1)^2 = -9$$

$$(z - 1) = \pm 3i$$

$$z = 1 \pm 3i$$

Practice:

Solve each of these quadratic equations.

a $x^2 + 5x + 7 = 0$ **b** $x^2 - 3x + 5 = 0$

c $2x^2 + 7x + 7 = 0$ **d** $3x^2 - 10x + 9 = 0$

TLMaths:

Quadratic Equations with Real Coefficients: https://www.youtube.com/watch?v=yfySgUaFbNU

Complex Conjugates: https://www.youtube.com/watch?v=Os3urOI8798&t=6s

Complex Conjugate Roots

Did you notice that all the complex numbers you found – being the roots of real equations – came up in complex conjugate pairs?

This is always the case if the original equation has real coeffeicients. We can show this using the quadratic formula on the equation $az^2 + bz + c = 0$, where a, b and c are all real numbers. The roots of this are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Since the $-\frac{b}{2a}$ is real, if the roots are complex, only the imaginary part is after the " $\pm$ ".

This means that if we know one complex root of a quadratic equation, we know the other root must be its complex conjugate.

Example:

Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients,

(a) state the value of the other root, β. (b) find the quadratic equation.

(a) As the equation has real coefficients, α and β are complex conjugates. $\beta = 7 - 2i$.

(b) $(z - \alpha)(z - \beta) = 0$

$(z - 7 + 2i)(z - 7 - 2i) = 0$

$z^2 - 7z - 7z + 49 - 4i^2 = 0$

$z^2 - 14z + 49 + 4 = 0$

$z^2 - 14z + 53 = 0$

1) The factorised form works just as well for complex equations as it does with real equations.

2) Expand and simplify. Since we know the equation is real, we don't need to work out all the imaginary parts (they will cancel out anyway!)

Practice:

A quadratic equation has a solution of $z = \sqrt{3} + i$

a Write down the other solution of the equation.

b Find a possible equation.

Find a quadratic equation where one solution is given as

a $2+i$ **b** $4-3i$ **c** $7i-1$

d $-5-2i$ **e** $a+3i$ **f** $5-bi$

For Your Interest:

This document introduces the world of complex numbers, a huge topic which has a great deal of fascinating applications. For an idea of some of them, consider watching videos on:

The Mandlebrot Set - https://www.youtube.com/watch?v=FFftmWSzgmk

Cardano's 'Useless numbers' - https://www.youtube.com/watch?v=_qvp9a1x2UM

Matrices

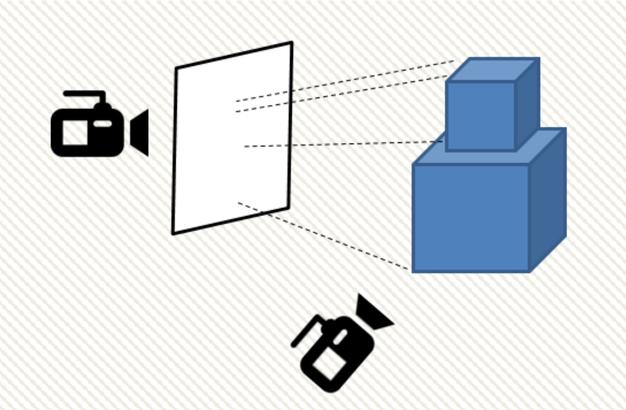
Introduction

A matrix (plural: matrices) is **simply an 'array' of numbers**, e.g. $\begin{pmatrix} 1 & 0 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

On a simple level, a matrix is just a way to organise values into rows and columns, and represent these multiple values as a single structure.

But the power of matrices comes from them **representing linear transformations/functions** (which we will particularly see in Chapter 7). We can

1. **Represent linear transformations** using matrices (e.g. rotations, reflections and enlargements)
2. Use them to **solve linear simultaneous equations**.



Matrices are particularly useful in 3D graphics, as matrices can be used to carry out rotations/enlargements (useful for changing the camera angle) or project into a 2D 'viewing' plane.

Dimensions of a Matrix

The dimension of a matrix is its **size**, in terms of its number of **rows** and **columns** (in that order).

Matrix	Dimensions
$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$	2×3
$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$	3×1
$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$	1×3

TLMaths - Introducing Matrices

TLMaths – Special Matrices

Adding and subtracting matrices

If two matrices have the same dimensions, they can be added or subtracted. We do this by adding and subtracting the individual **elements** (numbers) within the matrices. This is exactly the same as what you do when adding or subtracting vectors – because vectors are matrices too!

Examples:
$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3-q & 3 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

TLMaths: Adding and Subrtracting Matrices

We can add multiple copies of the same matric together in the same way – meaning we can multiply matrices by scalars.

Examples:
$$3\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 9 & -21 \\ 12 & 0 & 15 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} \qquad 2\boldsymbol{A} = \begin{pmatrix} 2q & -6 \\ 2 & 2 \\ -8 & 2 \end{pmatrix}$$

TLMaths: Multiplying Matrices by a Scalar

Practice:

Calculate

a $\begin{pmatrix} 5 & -1 & 2 \\ 0 & 6 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 4 & 0 \\ 5 & -7 & 2 \end{pmatrix}$

b $\begin{pmatrix} 4 & -2 \\ 8 & -5 \end{pmatrix} - \begin{pmatrix} 8 & 4 \\ 7 & -2 \end{pmatrix}$

c $4\begin{pmatrix} -1 & 0 & 4 \\ 2 & 5 & -3 \\ 8 & -6 & 0 \end{pmatrix}$

d $\begin{pmatrix} -6 & 14 \\ 3 & 9 \\ 2 & -4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 & -4 \\ 6 & 3 \\ -2 & 0 \end{pmatrix}$

If $\mathbf{A} = \begin{pmatrix} 9 & -4 \\ 0 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 & 4 \\ 2 & -6 \\ -2 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -5 & -2 \\ 7 & 0 \\ 3 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$

Calculate if possible or, if not, explain why.

i $\mathbf{A} + \mathbf{D}$ **ii** $\mathbf{A} + \mathbf{B}$

iii $\mathbf{B} - \mathbf{C}$ **iv** $3\mathbf{B}$

<u>Matrix Multiplication</u>

Unlike addition of matrices, matrix multiplication is a wholly new skill that we examine in Year 12 Further Maths.

To multiply two matrices, you need to multiply rows of the first matrix by columns of the second matrix, summing all the individual products together to make each new element for the new matrix. For a detailed explanation, watch and take notes from TLMaths: Multiplying Matrices.

Practice:

$\begin{pmatrix} -2 & 1 \\ 0 & 9 \\ -5 & 0 \end{pmatrix}\begin{pmatrix} -6 & 3 \\ 4 & 8 \end{pmatrix}$

$\begin{pmatrix} 5 & -3 & 0 \\ 2 & -4 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -5 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 9 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 4 & 7 & -4 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 3 \end{pmatrix}\begin{pmatrix} -1 & 2 & 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & -1 \\ -2 & 0 & 2 \\ 0 & -3 & 1 \end{pmatrix}^2$

The dimensions of each matrix have to meet a requirement before the multiplication can be done – the size of the rows in the first matrix must match the size of the columns in the second matrix. The product will have the number of rows in the first, and the number of columns from the second.

Dimensions of A	Dimension of B	Dimensions of AB (if valid)
2×3	3×4	2×4
1×3	2×3	Not valid.
6×2	2×4	6×4
1×3	3×1	1×1
7×5	7×5	Not valid.
10×10	10×9	10×9
3×3	3×3	3×3

Valid if these match

Comes from the 'outside numbers'.

A matrix can only be raised to a power if it is a **square matrix.**

Practice:

If $\mathbf{A} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$,

$\mathbf{C} = \begin{pmatrix} 2 & 9 & 4 \\ -3 & 0 & -5 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$,

find, if possible, or if the calculation is not possible, explain why.

a **AB** b **CD** c **BCD**

d **CB** e $\mathbf{B}^2\mathbf{A}$ f $\mathbf{C}^2$

<u>Commutivity and Associativity</u>

An operation is **commutative** if the elements in it can be written in a different order. For example, with numbers, addition and multiplication are commutative:

$$3 + 7 = 7 + 3 \qquad\qquad 4 \times -2 = -2 \times 4$$

But subtraction is not:

$$12 - 8 = 4$$

$$8 - 12 = -4.$$

Matrix multiplication, unlike multiplication with numbers, is **not commutative**. Before watching the below video, can you show why not?

[TLMaths: Matrix Multiplication is Not Commutative.]

An operation is **associative** if the operations within an expression can be done in any order – i.e. 'you can move the brackets'.

For example, multiplication is associative:

$$(7 \times 2) \times 3 = 42$$

$$7 \times (2 \times 3) = 42$$

But subtraction is not:

$$(7 - 2) - 3 = 2$$

$$7 - (2 - 3) = 8$$

Watch and make notes on the following videos exploring this property of matrices:

Is Matrix Multiplication is Associative?

Proving Matrix Multiplication is Associative

You are not required to memorise this proof, but the associativity of matrix multiplication is an important part of the working you will be able to do with Matrices in Further Maths, and is an important step in being able to write out matrix proofs.

Matrices for 2D Transformations

Since a coordinate can be written as a 2x1 vector ('position vector') $\begin{pmatrix} x \\ y \end{pmatrix}$, we can **multiply** this vector by a 2x2 matrix to transform it into another position vector, i.e. a coordinate.

Example: $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$. So the matrix $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ **transforms** the point $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

TLMaths: Transforming Coordinates

Practice: A square has coordinates $(1,1)$, $(3,1)$, $(3,3)$ and $(1,3)$. Find the vertices of the image of S under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Sketch S and the image of S on a coordinate grid.

More generally, transforming a point $\begin{pmatrix} x \\ y \end{pmatrix}$ simply involves multiplying it by some matrix. From above we can see that multiplying by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the mapping $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. We will see how we can use certain matrices to represent certain well-known transformations, e.g. $(x, y) \to (3x, 3y)$, i.e. an enlargement of scale factor 3 centred about the origin.

Stretches and Enlargements

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

The x and y values are both being scaled by 3, so this represents an **enlargement scale factor 3 about the origin**.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$$

The x value is scaled but not the y. This represents a **stretch scale factor 2 parallel to the x-axis** (but NOT an enlargement).

✏ $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ represents a stretch scale factor a parallel to the x-axis and a stretch scale factor b parallel to the y-axis. When $a = b$ this represents an enlargement.

TLMaths: Stretches and Enlargements

Reflections and Rotations

Reflection and rotation matrices have specific standard forms that help you to recognise them. Watch and take notes on this series of TLMaths videos to show you where these matrices come from and how to use them to carry out transformations.

Deriving the Reflection Matrix

Finding a Particular Reflection Matrix

Reflections

Rotations

Deriving the Rotation Matrix

Examples of Finding a Rotation Matrix

[Describing a Rotation Matrix]

And finally – what is a linear transformation anyway?

$ax + by$ is known as a **linear combination** of x and y. **Each row** of the matrix we're multiplying by provides an instruction of how to generate each number of the new coordinate. e.g. given $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + y \end{pmatrix}$, the new x value is $2x + 3y$ and the new y value is $x + y$, i.e. linear combinations of the old x and y values.

Notice that through all linear transformations the origin is preserved – the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ stays at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. No combination of 0s will ever be anything other than 0!

This means that **translations** (which do not preserve the origin) **cannot** be represented by matrix multiplication.

For Your Interest

Matrices and matrix multiplication is another big topic in Further Maths, expanding into three dimensions and connected to other topics like solving simultaneous equations, the equations of lines and 2D planes, and statistics. Outside of A Level, it enables you to develop in many other sciences, most notably computer programming. Consider watching the following videos on:

The True Power of the Matrix (Transformations in Graphics)
Math and Movies (Animation at Pixar)