Solihull Sixth Form College

GCSE to A-Level Maths Bridging the Gap

Most of the work covered in A-level Mathematics is focused on algebra. This document is designed to support you with many key skills at GCSE that will enable you cope with the initial demands of the A level Mathematics course.
This includes expanding brackets, factorising, algebraic fractions, indices, rearranging formulae, rearranging formulae, solving other equations such as quadratics, straight line graphs, surds and trigonometry.

In addition, the following link
https://sites.google.com/view/tlmaths/home/gcse-to-a-level-maths-bridging-the-gap
is a set of 19 videos (Jack Brown) that go through examples in line with the sections shown below.

Section 1: Multiplying two brackets

Key point: When multiplying two brackets, remember FOIL. This is short for First, Outside, Inside, Last, and helps you remember which numbers to multiply.

Example 1: Expand $(a+2)(a+5)$

Solution: First: $a \times a = a^2$
Outside: $a \times 5 = 5a$
Inside: $2 \times a = 2a$
Last: $2 \times 5 = 10$

Add these up to give the solution: $a^2 + 5a + 2a + 10 = a^2 + 7a + 10$

Example 2: Expand $(m-5)(m+4)$

Solution: First: $m \times m = m^2$
Outside: $m \times 4 = 4m$
Inside: $^{-}5 \times m = -5m$
Last: $^{-}5 \times 4 = ^{-}20$

Add these up to give the solution: $m^2 + 4m - 5m - 20 = m^2 - m - 20$

Exercise 1A

Multiply out these brackets

1. $(x+2)(x+3)$
2. $(x-5)(x-6)$
3. $(a+2)(a+1)$
4. $(x-9)(x-3)$
5. $(x+10)(x-1)$
6. $(5x-1)(2x-4)$
7. $(2x-5)(3x-2)$
8. $(3x+y)(4x+y)$
9. $(2g+4)(2g-4)$
10. $(3p+2q)(3p-q)$

Key point: When squaring a bracket, make sure you write them both out so you do not forget all four terms.

Example 3: Expand $(x+2)^2$

Solution: $(x+2)^2 = (x+2)(x+2)$

First: $x \times x = x^2$

Outside: $x \times 2 = 2x$

Inside: $2 \times x = 2x$

Last: $2 \times 2 = 4$

Add these up to give the solution: $x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

Exercise 1B

Square these brackets

1. $(x+3)^2$
2. $(x-2)^2$
3. $(x+8)^2$
4. $(2x+3)^2$
5. $(3x+2y)^2$

Section 2a: Factorising with algebraic expressions

Key point: A highest common factor is an expression that is contained in every term of the original expression. When you have found the common factor, it can then be taken out the front of a bracket.

Example 1: Factorise $4p+6$

Solution: The common factor is 2, because $4p = 2 \times 2p$ and $6 = 2 \times 3$. So $4p+6 = 2\times 2p + 2\times 3 = 2(2p+3)$

Example 2: Factorise $15ab+10a$

Solution: The highest common factor is $5a$, because $15ab = 5a \times 3b$ and $10a = 5a \times 2$. So $15ab + 10a = 5a\times 3b + 5a \times 2 = 5a(3b+2)$

Exercise 2A

Factorise these expressions

1. $3x-12$
2. $3a+5a^2$
3. $2ab-6ac$
4. $5a^2b+10ab^2$
5. $12x-6y+8z$
6. $14a^2-7a^3$
7. $15xy-5y$
8. $3a^2b-6a^3b$
9. $10z^3-25z^2+15z$
10. $7a^3b^3c^3-14a^2b^3c^4$

Solihull Sixth Form College

Section 2b: Factorising a quadratic expression

Key point: Difference of two squares; A quadratic that is of the form $x^2 - b^2$ can be factorised to give $(x-b)(x+b)$

Example 1: Factorise $x^2 - 25$

Solution: $x^2 - 25 = x^2 - 5^2 = (x-5)(x+5)$

Example 2: Factorise $9x^2 - 4y^2$

Solution: $9x^2 - 4y^2 = (3x)^2 - (2y)^2 = (3x-2y)(3x+2y)$

Example 3: Factorise $8x^2 - 200$

Solution: To get square numbers in front of the x and after the $-$ symbol, we need to take out a factor of 2; so $8x^2 - 200 = 2(4x^2 - 100) = 2\left((2x)^2 - 10^2\right) = 2(2x-10)(2x+10)$

Exercise 2B

Factorise the following expressions

1. $x^2 - 4$
2. $9 - 16y^2$
3. $1 - 49t^2$
4. $100x^2 - 1$
5. $121x^2 - 144y^2$
6. $81p^2 - 64q^2$
7. $8 - 2x^2$
8. $7a^2 - 63b^2$
9. $3x^2 - 108$
10. $7x^2 - 343y^2$

Key point; To factorise an expression of the type $x^2 + bx + c$, you need to find two numbers which add together to give b and multiply together to give c.

Example 4: Factorise $x^2 + 7x + 12$

Solution: The two numbers which add to give 7, and multiply to give 12 are 3 and 4. So the brackets are given by $x^2 + 7x + 12 = (x+3)(x+4)$

Example 5: Factorise $x^2 - 3x + 2$

Solution: The two numbers which add to give -3 and multiply to give 2 are -1 and -2. So the brackets are given by $x^2 - 3x + 2 = (x-1)(x-2)$

Example 6: Factorise $x^2 - 3x - 10$

Solution: the two numbers which add to give -3 and multiply to give -10 are 2 and -5. So, the brackets are given by $x^2 - 3x - 10 = (x+2)(x-5)$

Solihull Sixth Form College

Exercise 2C

Factorise the following expressions

1. x^2+5x+6
2. $x^2+7x+10$
3. x^2+2x+1
4. x^2-7x+6
5. $x^2-12x+27$
6. $x^2-9x+20$
7. x^2-2x-8
8. x^2+2x-3
9. $x^2+4x-32$
10. x^2+x-20

Key point: To factorise an expression of the form ax^2+bx+c, you need to work out the factors of a and put them before the x's in brackets. Then work out the factors of c and try all possible combinations of brackets to find the correct one. NB if c is positive, both brackets have the same sign as b. If c is negative, the brackets have different signs.

Example 7: Factorise the quadratic $3x^2+16x+5$

Solution: The factors of 3 are 1 and 3. So, the brackets are of the form $(3x+\_)(x+\_)$ (+ signs as c is positive and b has a + sign). The factors of 5 are 1 and 5. So the two choices of brackets are

$$(3x+5)(x+1) \quad \text{Or} \quad (3x+1)(x+5)$$

Multiplying these both out give

$$(3x+5)(x+1)=3x^2+8x+5 \quad \text{Or} \quad (3x+1)(x+5)=3x^2+16x+5$$

The second choice gives the quadratic we need, so the answer is $(3x+1)(x+5)$

Example 8: Factorise the quadratic $2x^2-7x+3$

Solution: The factors of 2 are 1 and 2. So the brackets are of the form $(2x-\_)(x-\_)$ (- signs as c is positive and b has a - sign). The factors of 3 are 1 and 3. So the two choices of brackets are

$$(2x-1)(x-3) \quad \text{Or} \quad (2x-3)(x-1)$$

Multiplying these out gives

$$(2x-1)(x-3)=2x^2-7x+3 \quad \text{Or} \quad (2x-3)(x-1)=2x^2-5x+3$$

The first choice gives the quadratic we need, so the answer is $(2x-1)(x-3)$

Example 9: Factorising the quadratic $6x^2+17x+7$

Solution:

$6x^2+3x+14x+7$

$3x(2x+1)+7(2x+1)$

$(2x+1)(3x+7)$

Solihull Sixth Form College

Exercise 2D

Factorise the following expressions;

1. $3x^2+7x+2$
2. $2x^2+3x+1$
3. $2x^2+5x-3$
4. $6x^2 +17x +5$
5. $12x^2 +x -6$
6. $6x^2-11x+3$

Section 3: Cancelling algebraic fractions

Key point: You can only cancel something if it multiplies BOTH the top and the bottom of the fraction.

Example 1: Simplify $\frac{36ab^2c^2}{6a^2bc^2}$.

Solution: The top and the bottom of the fraction both have a factor of $6abc^2$, so this can be cancelled.
Taking it from the top leaves $6b$
($36\div 6=6$, the a cancels, one of the b's cancel and the c^2 cancels)
From the bottom leaves a
($6\div 6=1$, one of the a's cancel, the b cancels and the c^2 cancels).
This means $\frac{36ab^2c^2}{6a^2bc^2}=\frac{6b}{a}$

Example 2: Simplify $\frac{x^2+x}{x^2-1}$

Solution: This needs to be factorised first; $x^2+x=x(x+1)$ and $x^2-1=(x+1)(x-1)$. So,
$\frac{x^2+x}{x^2-1}=\frac{x(x+1)}{(x+1)(x-1)}$. Both top and bottom are multiplied by $x+1$, so the $x+1$'s cancel. This leaves $\frac{x}{x-1}$.

Exercise 3A

Simplify the following expressions

1. $\frac{15a^2b^2}{6ab}$
2. $\frac{12a^3b^2c}{4a^2b^2c^2}$
3. $\frac{3x^2yz}{6xy^2z^2}$
4. $\frac{2x}{x^2-3x}$
5. $\frac{5x^2-20x}{10x^2}$
6. $\frac{x^2+2x+1}{x^2-1}$
7. $\frac{x^2-5x+4}{x^2-2x-8}$

Solihull Sixth Form College

Section 4: Laws of indices

Key point: The laws of indices are;

Multiplicative laws; 1) $n^a \times n^b = n^{a+b}$, 2) $n^a \div n^b = n^{a-b}$

Use with brackets; 3) $\left(n^a\right)^b = n^{a\times b}$, 4) $(n\times m)^a = n^a \times m^a$, 5) $\left(\frac{n}{m}\right)^a = \frac{n^a}{m^a}$

Negative powers; 6) $n^{-a} = \frac{1}{n^a}$, 7) $\left(\frac{n}{m}\right)^{-a} = \frac{m^a}{n^a}$

Fractional powers; 8) $n^{\frac{1}{b}} = \sqrt[b]{n}$, 9) $n^{\frac{a}{b}} = \sqrt[b]{n^a}$ or $\left(\sqrt[b]{n}\right)^a$

Other laws; 10) $n^0 = 1$ 11) $n^1 = n$

Example 1: The following expressions demonstrate the laws of indices;

$3^2 \times 3^3 = 3^5$, $5^5 \div 5^2 = 5^3$

$\left(2^2\right)^4 = 2^8$, $(2\times 3)^2 = 2^2 \times 3^2$, $\left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$

$5^{-7} = \frac{1}{5^7}$, $\left(\frac{5}{7}\right)^{-3} = \frac{7^3}{5^3}$

$8^{\frac{1}{2}} = \sqrt{8}$ $9^{\frac{1}{5}} = \sqrt[5]{9}$ $6^{\frac{3}{2}} = \sqrt{6^3}$ or $\left(\sqrt{6}\right)^3$

Example 2: Work out the following $64^{\frac{-1}{2}} \times 2^2$

Solution: Calculating the values separately before we multiply them gives $64^{\frac{-1}{2}} = \frac{1}{64^{\frac{1}{2}}} = \frac{1}{\sqrt{64}} = \frac{1}{8}$ and $2^2 = 4$. Therefore $64^{\frac{-1}{2}} \times 2^2 = \frac{1}{8} \times 4 = \frac{4}{8} = \frac{1}{2}$

Exercise 4A

Work out these, giving your answers as fractions or whole numbers

1. 4^{-1}
2. $8^{\frac{1}{3}}$
3. 9^0
4. $27^{\frac{4}{3}}$
5. $64^{\frac{1}{2}}$
6. $16^{\frac{-1}{4}}$
7. $2^2 \times 9^{\frac{1}{2}}$
8. $25^{\frac{3}{2}}$
9. $2^2 + 3^0 + 16^{\frac{1}{2}}$
10. $5^{-2} \times 10^5 \times 16^{\frac{-1}{2}}$

Example 3: Rewrite the following as a power of 2; $\sqrt[5]{64}\times\frac{1}{2}$

Solution: Starting with the 64, we know that $64=2^6$. Secondly, law 9) of the laws of indices tells us that $\sqrt[5]{2^6}=2^{\frac{6}{5}}$. Starting with the $\frac{1}{2}$, law 6) tells us that $\frac{1}{2}=2^{-1}$. Finally, combining these we have $2^{\frac{6}{5}}\times2^{-1}$.

Using law 1) gives $2^{\frac{6}{5}-1}=2^{\frac{1}{5}}$. This gives the power of two as $2^{\frac{1}{5}}$

Exercise 4B

Write these numbers as a power of the number in brackets

1. 27 (3)
2. 81^2 (3)
3. 32 (2)
4. 0.25 (2)
5. 5^4-5^3 (5)

Section 5: Rearranging formulae

Key point: When you have the subject of an equation on the bottom of a fraction, you should multiply the whole expression by the bottom of the fraction to move the subject to the top. If the subject appears more than once in a formula, collect the subject to the one side, factorise and then divide out the unwanted factor.

Example 1: Make x the subject of the following formula; $u=v+\frac{w}{x+1}$

Solution: x appears on the bottom of a fraction, so every thing is multiplied by the bottom;

$u(x+1)=v(x+1)+w$

Multiplying out these brackets gives $ux+u=vx+v+w$

x appears more than once in this formula, so we move all the x's to one side;

$ux-vx=v+w-u$

Factorising gives $x(u-v)=v+w-u$

Then dividing $x=\frac{v+w-u}{u-v}$

Example 2: Make p the subject of the formula; $2+s=\sqrt{\frac{2}{p}}$

Using the laws of indices gives $2+s=\frac{\sqrt{2}}{\sqrt{p}}$

p appears on the bottom of a fraction, so multiply everything

$2\sqrt{p}+s\sqrt{p}=\sqrt{2}$

Factorising gives $\sqrt{p}(s+2)=\sqrt{2}$

Dividing gives $\sqrt{p}=\frac{\sqrt{2}}{s+2}$

Lastly, we require p, so we square everything, giving $p=\frac{2}{(s+2)^2}$

Exericse 5A:

Rearrange these formulae making the letter in the brackets the subject;

1. $s = at + 2bt$ (t)
2. $s = \frac{1}{a} + b$ (a)
3. $s - 2ax = b(x - s)$ (x)
4. $\frac{a}{b} - 2a = b$ (a)
5. $a = b + c^2$ (c)

Section 6: Simultaneous equations

Example :

Solve the following pairs of simultaneous equations

1) $7x + 2y = 32$
$x + y = 1$

2) $5x + 2y = 26$
$4x - 3y = 7$

Solution 1)

Double the second equation to give

$7x + 2y = 32$

$2x + 2y = 2$

Subtract the new second equation from the new first, and solve the resulting equation to find x

$5x = 30$

$x = 6$

Substitute into either of the original equations to find y

$x + y = 1$

$\Rightarrow \quad 6 + y = 1$

$y = -5$

Solution 2)

Multiply the first equation by 3 and the second equation by 2 to give

$15x + 6y = 78$

$8x - 6y = 14$

Add the two equations and solve

$23x = 92$

$x = 4$

Substitute into either of the original equations to find y

$5x + 2y = 26$

$\Rightarrow 20 + 2y = 26$

$2y = 6$

$y = 3$

Exercise 6A: Solve these pairs of simultaneous equations

a) 3e + 2f = 28
2e + 7f = 47

b) 2a + 5b = 52
3a + 2b = 34

c) 9x + 8y = 1
2x + 3y = -1

d) 6c + 4d = 48
5c + 3d = 38

e) 2x + 7y = 12
5x + 3y = 1

f) 8u + 7v = 75
6u + 8v = 70

g) 3s + 2t = 57
2s + 5t = 82

h) 4x + 2y = 0
9x + 5y = 2

Solihull Sixth Form College

Section 7: Solving Linear Equations

Example 1:

Solve the following equations 1) $5x+4=11$ and 2) $7(x-2)=7$

Solution 1)

$5x+4=11$

$5x=11-4$

$5x=7$

$x=7\div 5=\frac{7}{5}$

Solution 2)

$7(x-2)=7$

$7x-14=7$

$7x=7+14=21$

$x=21\div 7=3$

Exercise 7A: Solve the following equations

1. $5x+7=32$
2. $2x+8=13$
3. $4x+3=8$
4. $10x+2=16$
5. $4(2x-1)=28$
6. $3(3x+4)=21$
7. $2(2x-7)=7$
8. $3(6x+4)=39$
9. $5x+7+2=12+3-3x$
10. $7x+6-5x+2=11$
11. $12x+7+4=6x+17+14+x$
12. $8x-3x-4=4-7x+4$
13. $7x+3x-6x+7=5x+13-4x$
14. $4(4x+1)=3(5x+4)$
15. $3(2x+1)+2(4x+2)=21$
16. $4(7x+2)=5(5x+6)$
17. $3(2x+1)=2(2x+7)$
18. $2(4x+3)+3(2x+1)=23$
19. $8(x+1)+5(2x+3)=4(4x+9)$
20. $2(2x+1)+3(4x+5)=57$

Section 8: Solving Quadratic equations

By Factorising

Key point: To solve quadratic equations, first rearrange to give $ax^2+bx+c=0$, then factorise. Finally, make each bracket equal to zero, and solve these two brackets for x.

Example 1: Solve the equation $2x^2-7x+3=0$

Solution: The factors of 2 are 1 and 2. So the brackets are of the form $(2x-\_)(x-\_)$ (- signs as c is positive and b has a - sign). The factors of 3 are 1 and 3. So the two choices of brackets are

$$(2x-1)(x-3) \qquad \text{Or} \qquad (2x-3)(x-1)$$

Multiplying these out gives

$$(2x-1)(x-3)=2x^2-7x+3 \qquad \text{Or} \qquad (2x-3)(x-1)=2x^2-5x+3$$

The first choice gives the quadratic we need, so the answer is $(2x\text{-}1)(x-3)$

Putting these two brackets equal to zero gives $2x-1=0$ and $x-3=0$. Solving these two equations gives $x=\frac{1}{2}$ and $x=3$.

Example 2: Solve the equation $x^2-16=0$

Solution: Spotting that this is a difference of two squares expression gives $x^2-16=x^2-4^2=(x-4)(x+4)$

Putting the two brackets equal to zero gives $x-4=0$ and $x+4=0$. Solving these two equations gives $x=4$ and $x=-4$

Exercise 8A

Solving these quadratic equations;

1. $2x^2-5x+12=0$
2. $2x^2-3x-5=0$
3. $2x^2-13x+15=0$
4. $x^2+x-30=0$
5. $x^2-4x+3=0$

By using the Quadratic Formula

Key point: To solve a quadratic equation, you must first rearrange it into the form $ax^2+bx+c=0$. Then you can follow one of three routes;

Completing the square, factorising or using the quadratic equation $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example 1: Solve the quadratic equation $x^2-3=2x$

Solution: First rearrange this equation to give $x^2-2x-3=0$. This has values $a=1$, $b=-2$ and $c=-3$. Putting these into the quadratic equation gives;

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times(-3)}}{2\times1}=\frac{2\pm\sqrt{4+12}}{2}=\frac{2\pm\sqrt{16}}{2}=\frac{2\pm4}{2}=\frac{6}{2}\text{ or }\frac{-2}{2}=3\text{ or }-1$$

Exercise 8B

Solve the following equations using the quadratic formula

1. $x^2 - 6x + 4 = 0$
2. $x^2 + 3x - 1 = 0$
3. $x^2 - 2 = 2x$
4. $x^2 + 8x + 3 = 0$
5. $x^2 + 2 = 5x$
6. $4x^2 + 3x - 2 = 0$
7. $3x^2 - 4x = 5$
8. $2x^2 - 5x - 2 = 0$

By completing the square

Key point: To complete the square of a quadratic equation, you need to start off by dividing through by the x^2-coefficient, giving $x^2 + Bx + C = 0$

This is then put into completed square form using $\left(x + \frac{1}{2}B\right)^2 = \frac{1}{4}B^2 - C$

To solve this, you square root each side and then solve the linear equations.

Example 1:

$x^2 - 6x + 4$

= $(x-3)^2 - 9$

= $(x-3)^2 - 5$

Example 2: Using completing square to solve $2x^2 + 10x + 5 = 0$

Solution: Dividing by a gives $x^2 + 5x + 2.5 = 0$

Using the completed square formula gives $\left(x + \frac{1}{2}\times 5\right)^2 = \frac{1}{4}\times 5^2 - 2.5$

i.e. $(x + 2.5)^2 = 6.25 - 2.5$ i.e. $(x + 2.5)^2 = 3.75$

Square rooting this gives $x + 2.5 = \pm\sqrt{3.75}$

So, $x = \pm\sqrt{3.75} - 2.5 = -0.56 \text{ or } -4.44$

Exercise 8C

Use the completing square method to confirm your answers to Exercise 8A

Section 9: Straight line graphs

Key point: A straight line graph has equation $y = mx + c$, where m is the gradient of the line and c is the value of the y-intercept of the line (i.e. the line passes through the point $(0, c)$).

Example 1: What is the equation of the line which has gradient 3 and passes through $(0,2)$?

Solution: $y = mx + c$, therefore $y = 3x + 2$

Exercise 9A

What are the equations of the following lines;

1. gradient is -1, passing through the point $(0,4)$
2. gradient is 5, passing through the point $(0,0)$
3. gradient is 4, passing though the point $(0,-1)$
4. gradient is -2, passing through the point $(0,5)$
5. gradient is 3, passing through the origin.

Key point: to work out the gradient of a line which passes through the points (a,b) and (c,d), you need to work out $m=\frac{d-b}{c-a}$.

Example 2: What is the gradient of the line which passes through the points $(2,10)$ and $(0,2)$? What is the equation of the line?

Solution: $(a,b)=(2,10)$ and $(c,d)=(0,2)$. Therefore $m=\frac{2-10}{0-2}=\frac{-8}{-2}=4$

The gradient is 4, and it passes through the point $(0,2)$, so $y=4x+2$

Exercise 9B

What is the gradient of the line passing through the following pairs of points? What is the equation of each line?

1. $(0,4)$ and $(6,6)$
2. $(0,0)$ and $(5,10)$
3. $(0,3)$ and $(2,7)$
4. origin and $(3,12)$
5. $(0,1)$ and $(4,3)$
6. $(0,5)$ and $(5,0)$
7. $(0,1)$ and $(-2,4)$
8. $(0,-3)$ and $(-6,0)$

Key point: The gradients of two parallel lines are equal. The gradients of two perpendicular (at right angles to each other) lines multiply to give -1.

Example 3: Give a possible equation for a line which is parallel to the line $y=2x$. Give a possible equation for a line which is perpendicular to $y=2x$.

Solution: A line which is parallel has equal gradient. So $y=2x+c$. As there is no restriction on c, we can chose any number. Possible answers include; $y=2x+1$, $y=2x+2$, $y=2x+3$ etc

A line which is perpendicular has gradient such that $2m=-1$, i.e. the product of the gradients is -1. So, $m=\frac{-1}{2}$. Again there is no restriction on c, so the possible answers are $y=\frac{-1}{2}x+1$, $y=\frac{-1}{2}x+2$, $y=\frac{-1}{2}x$, $y=\frac{-1}{2}x-1$ etc.

Exercise 9C

For each line below, give an equation of a line parallel and an equation for a line that is perpendicular.

1. $y=-2x$
2. $y=\frac{-1}{3}x+6$
3. $y=6x-3$
4. $y=0.5x+1$

Key point:
To find the midpoint of a line, you need to find the average of the x- and y-coordinates. Using algebra, the midpoint of the line connecting (a,b) and (c,d) is given by $\left(\frac{a+c}{2},\frac{b+d}{2}\right)$

Example 1: Find the mid-point of the line connecting the points $A(2,1)$ and $B(6,7)$

Solution: $\text{Midpoint} = \left(\frac{2+6}{2},\frac{1+7}{2}\right) = \left(\frac{8}{2},\frac{8}{2}\right) = (4,4)$

Example 2: Find the mid-point of the line connecting the points $C(-2,1)$ and $D(2,5)$

Solution: $\text{Midpoint} = \left(\frac{-2+2}{2},\frac{1+5}{2}\right) = \left(\frac{0}{2},\frac{6}{2}\right) = (0,3)$

Exercise 9D
Find the coordinates of the mid-point of the line connecting each of the following pairs of coordinates:

1) $A(1,4)$ and $B(1,8)$
2) $C(1,5)$ and $D(7,3)$
3) $E(2,3)$ and $F(8,6)$
4) $G(3,7)$ and $H(8,2)$
5) $I(-2,3)$ and $J(4,1)$
6) $K(-4,-3)$ and $L(-6,-11)$

Section 10: Inequalities

Key point:
Statements involving the signs $<$, $>$, $\leq$ or $\geq$ are called inequalities. They are solved in the same way as equations, BUT if you multiply or divide by a negative number, you must reverse the direction of the inequality.

Example 1: Solve this inequality; $3x+4<10$

$3x+4<10$

Solution:
$3x<6$ subtract 4 from each side
$x<2$ divide each side by 3

Example 2: Solve the inequality; $2x-5\leq 4-3x$

$2x-5\leq 4-3x$

Solution:
$2x\leq 9-3x$ Add 5 to each side
$5x\leq 9$ Add $3x$ to each side
$x\leq 1.8$ Divide each side by 5

Example 3: Solve the inequality; $x+4>3x-2$

Solution:

$$x+4>3x-2$$

$x>3x-6$ Subtract 4 from each side

$-2x>-6$ Subtract $3x$ from each side

$x<3$ Divide each side by -2; reverse the sign

Exercise 10A

Solve the following inequalities;

1. $x-3\le 4$
2. $x+7>9$
3. $2x-3<5$
4. $3x+4\le 7$
5. $3x\ge x-2$
6. $5x>3-x$
7. $4x>2x+5$
8. $3x-6\ge x+2$
9. $2x-7\ge 5x+8$
10. $2x+9>4x+5$

Example 4: Solve the inequality $2(3x-1)>4x+7$

Solution:

$$2(3x-1)>4x+7$$

$6x-2>4x+7$ Multiply out brackets

$6x>4x+9$ Add 2

$2x>9$ Subtract $4x$

$x>4.5$ Divide by 2

Exercise 10B

Solve these inequalities

1. $2x+3<5$
2. $3(2x-1)>15$
3. 2x+6 < x +3
4. $2(3x-1)\ge 4x+6$
5. $3(2x-4)<5(x-6)$

Section 11: Surds

Key point A surd is an expression involving a square root symbol. To manipulate them, you need to keep in mind the following rule; $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$. To simplify a surd, you need to find a factor which is a square number. This is then separated using the above rule, and square rooted.

Example 1: Simplify a) $\sqrt{50}$ and b) $\sqrt{72}$
Solution:
a) the number 50 has a factor of 25 (which is a square number)
So; $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5 \times \sqrt{2} = 5\sqrt{2}$
b) the number 72 had a factor of 36 (which is a square number)
So; $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$

Example 2: Simplify $\sqrt{12} \times \sqrt{27}$
Solution: $12 \times 27 = 324$. Using a calculator, this is a square number with $\sqrt{324} = 18$. So $\sqrt{12} \times \sqrt{27} = \sqrt{324} = 18$

Exercise 11A
Simplify each of these surds

1. $\sqrt{12}$
2. $\sqrt{1000}$
3. $\sqrt{45}$
4. $\sqrt{20} \times \sqrt{18}$
5. $\sqrt{20} \div \sqrt{5}$
6. $\sqrt{80} \times \sqrt{50}$

Key point: If you have an expression of the form $a + b\sqrt{\ }$, you should treat the a and the $b\sqrt{\ }$ separately, just as you would treat letters and numbers separately in algebra.

Example 3: If $x = 5 + \sqrt{2}$ and $y = 3 - \sqrt{2}$, what are $x + y$ and y^2?
Solution:
$x + y = 5 + \sqrt{2} + 3 - \sqrt{2} = (5+3) + (1-1)\sqrt{2} = 8 + 0\sqrt{2} = 8$
$y^2 = (3-\sqrt{2})^2 = (3-\sqrt{2})(3-\sqrt{2}) = 3 \times 3 - 3 \times \sqrt{2} - \sqrt{2} \times 3 + \sqrt{2} \times \sqrt{2}$ $= 9 - 3\sqrt{2} - 3\sqrt{2} + 2 = 11 - 6\sqrt{2}$

Exercise 11B
For $x = 4 + \sqrt{3}$, $y = 4 - \sqrt{3}$ and $z = 3\sqrt{3}$, simplify

1. $x + y$
2. $y - z$
3. $x + z$
4. x^2
5. $y \times z$
6. $x + y + z$

Key point: If you have a fraction with a surd denominator, you need to rationalise this. To do so, you multiply the top and bottom by the surd.

Example 4: Rationalise the denominator of the fraction $\frac{2}{\sqrt{3}}$

Solution: Multiply the top and bottom by $\sqrt{3}$ gives $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Example 5: Rationalise the denominator of $\frac{3}{4\sqrt{5}}$

Solution: Multiply the top and bottom by $\sqrt{5}$ giving $\frac{3\sqrt{5}}{4\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{4\times 5} = \frac{3\sqrt{5}}{20}$

Exercise 11C -Rationalise the denominators of the following fractions

1. $\frac{1}{\sqrt{2}}$
2. $\frac{5}{\sqrt{7}}$
3. $\frac{9}{\sqrt{20}}$
4. $\frac{3}{\sqrt{2}}$
5. $\frac{7}{\sqrt{50}}$
6. $\frac{6}{\sqrt{8}}$
7. $\frac{12}{\sqrt{75}}$
8. $\frac{10}{\sqrt{5}}$
9. $\frac{20}{\sqrt{32}}$
10. $\frac{4}{3\sqrt{10}}$

<u>Section 12: Trigonometry</u>

Key point:
To find the length of the hypotenuse h of a right-angled triangle, substitute the lengths of the shorter sides, a and b, into the formula $h = \sqrt{a^2 + b^2}$. NB This only works or a right-angled triangle.

Example 1: What is the length of the hypotenuse of a triangle when the shorter sides are $a = 9$ cm and $b = 12$ cm?
Solution: $h = \sqrt{a^2 + b^2} = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$ cm

Example 2: What is the length of the hypotenuse of a triangle when the shorter sides are $a = 1$ cm and $b = 2$ cm?
Solution: $h = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5} \approx 2.23$ cm

Exercise 12A
What is the length of the hypotenuse to 2dp when the shorter sides of a triangle are as follows:
1. $a = 3$ cm and $b = 4$ cm
2. $a = 5$ cm and $b = 10$ cm
3. $a = 2$ cm and $b = 5$ cm
4. $a = 8$ cm and $b = 8$ cm
5. $a = 7$ cm and $b = 8$ cm
6. $a = 12$ cm and $b = 5$ cm

Key point:
To find one of the short sides of a right-angled triangle, substitute the hypotenuse and the other short length into either $b = \sqrt{h^2 - a^2}$ or $a = \sqrt{h^2 - b^2}$

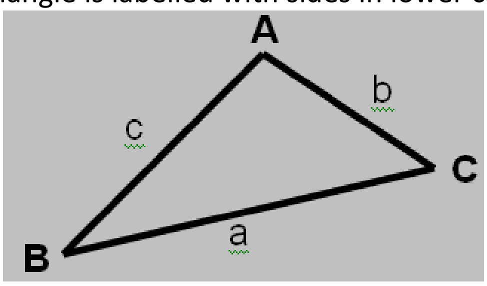
Example 3: What is the missing length of a triangle when the hypotenuse is $h = 8$ cm and one of the short sides is $b = 5$ cm?
Solution: $a = \sqrt{h^2 - b^2} = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39} \approx 6.24$ cm to 3sf

Example 4: What is the missing length of a triangle with $h = 12$ cm and $a = 3$ cm?
Solution: $b = \sqrt{h^2 - a^2} = \sqrt{12^2 - 3^2} = \sqrt{144 - 9} = \sqrt{135} \approx 11.6$ cm to 3sf

Exercise 12B
What is the missing length to 2 dp of a right angled triangle with the following sides?
1. $h = 17$ cm and $a = 15$ cm
2. $h = 9$ cm and $b = 7$ cm
3. $a = 12$ cm and $h = 20$ cm
4. $b = 8$ cm and $h = 30$ cm
5. $h = 169$ cm and $a = 5$ cm
6. $a = 3$ cm and $h = 5$ cm

Exercise 12C
What is the missing length to 2 dp of a right angled triangle with the following sides?
1. $a = 5$ cm and $b = 12$ cm
2. $a = 3$ cm and $b = 5$ cm
3. $a = 5$ cm and $h = 8$ cm
4. $h = 25$ cm and $b = 7$ cm
5. $a = 4.6$ cm and $b = 6.1$ cm
6. $a = 6.8$ cm and $h = 9.3$ cm

Key point: A general triangle is labelled with sides in lower case letters, and the opposite angles in CAPITAL LETTERS. For example;



NB; angle/side pairs are given the same letter, with one upper and the other lower case.

Key point: In any triangle the sine rule holds true; $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$. The sine rule can be used to work out a missing side length (if you know its opposite angle and another angle/side pair) OR it can work out a missing angle (if you know its opposite side and another angle/side pair).

Example 1: What is the missing side of a triangle with $B=52^\circ$, $C=40^\circ$ and $c=30$ cm?
Solution: Using the sine rule $\frac{b}{\sin 52}=\frac{30}{\sin 40}$.
Therefore $b=\frac{30\sin 52}{\sin 40}=\frac{30\times 0.788}{0.643}=36.778$ cm to 3dp

Example 2: What is the missing angle of a triangle with $a=9$ cm, $c=7.1$ cm and $A=35^\circ$?
Solution: Using the sine rule $\frac{9}{\sin 35}=\frac{7.1}{\sin C}$. Rearranging this gives $\sin C=\frac{7.1\sin 35}{9}$, and therefore

$$C=\sin^{-1}\left(\frac{7.1\sin 35}{9}\right)=\sin^{-1}\left(\frac{7.1\times 0.574}{9}\right)=\sin^{-1}(0.452)=26.9^\circ$$

Exercise 12D
Find the missing side or angle for the following triangles to 2dp
1. $B=86^\circ$, $C=48^\circ$ and $b=7.24$ cm
2. $P=112^\circ$, $Q=42^\circ$ and $q=8.4$ cm
3. $G=52^\circ$, $F=73^\circ$ and $f=17.7$ cm
4. $L=80^\circ$, $l=13.3$ cm and $m=12.8$ cm
5. $C=71^\circ$, $a=72$ m and $c=100$ m

Key point: If you know the length of two sides and the angle between them, you can work out the area of the triangle using $Area=\frac{1}{2}ab\sin C$

Example 3: Two sides of a triangle are $a=3$ cm and $b=6$ cm and the angle between these sides is $C=60^\circ$. What is the area of the triangle?
Solution: Substituting in the values for the a, b and C gives;
Area = $\frac{1}{2}\times 3\times 6\times sin60=7.79cm^2$ (3sf)

Exercise 12E
What is the area of these triangles to 2 dp
1. Sides are 2.4 cm and 8.6 cm; included angle is 60°
2. Sides are 3.1 cm and 3.8 cm; included angle is 62°
3. Sides are 8 cm and 5 cm; included angle is 100°
4. Sides are 7.5 cm and 8.1 cm; included angle is 42.4°
5. Sides are 6.5 cm and 7 cm; non-included angles are 81° and 67°

Solihull Sixth Form College

Answers to GCSE to A level

Ex 1A

1. x^2+5x+6
2. $x^2-11x+30$
3. a^2+3a+2
4. $x^2-12x+27$
5. $x^2+9x-10$
6. $10x^2-22x+4$
7. $6x^2-19x+10$
8. $12x^2+7xy+y^2$
9. $4g^2-16$
10. $9p^2+3pq-2q^2$

Ex 1B

1. x^2+6x+9
2. x^2-4x+4
3. $x^2+16x+64$
4. $4x^2+12x+9$
5. $9x^2+12xy+4y^2$

Ex 2A

1. $3(x-4)$
2. $a(3+5a)$
3. $2a(b-3c)$
4. $5ab(a+2b)$
5. $2(6x-3y+4z)$
6. $7a^2(2-a)$
7. $5y(3x-1)$
8. $3a^2b(1-2a)$
9. $5z(2z^2-5z+3)$
10. $7a^2b^3c^3(a-2c)$

Ex 2B

1. $(x+2)(x-2)$
2. $(3+4y)(3-4y)$
3. $(1+7t)(1-7t)$
4. $(10x+1)(10x-1)$
5. $(11x+12y)(11x-12y)$
6. $(9p+8q)(9p-8q)$
7. $2(2+x)(2-x)$

8. $7(a+3b)(a-3b)$
9. $3(x+6)(x-6)$
10. $7(x+7y)(x-7y)$

Ex 2C
1. $(x+2)(x+3)$
2. $(x+5)(x+2)$
3. $(x+1)(x+1)$
4. $(x-6)(x-1)$
5. $(x-3)(x-9)$
6. $(x-5)(x-4)$
7. $(x-4)(x+2)$
8. $(x-1)(x+3)$
9. $(x-4)(x+8)$
10. $(x-4)(x+5)$

Ex 2D
1. $(x+2)(3x+1)$
2. $(x+1)(2x+1)$
3. $(x+3)(2x-1)$
4. (3x+1)(2x+5)
5. (3x -2) (2x +3)
6. (2x-3)(3x-1)

Ex 3A
1. $\frac{5xy}{2}$
2. $\frac{3a}{c}$
3. $\frac{x}{2yz}$
4. $\frac{2}{x-3}$
5. $\frac{x-4}{2x}$
6. $\frac{x+1}{x-1}$
7. $\frac{x-1}{x+2}$

Ex 4A

Solihull Sixth Form College

1. $\frac{1}{4}$
2. 2
3. 1
4. 81
5. 8
6. $\frac{1}{2}$
7. 12
8. 125
9. 9
10. 1000

Ex 4B
1. 3^3
2. 3^8
3. 2^5
4. 2^{-2}
5. $2^2 \times 5^3$

Ex 5A
1. $t = \frac{s}{a+2b}$
2. $a = \frac{1}{s-b}$
3. $x = \frac{s+bs}{2a+b}$
4. $a = \frac{b^2}{1-2b}$
5. $c = \sqrt{a-b}$

Ex 6A

1a. e = 6 , f = 5

b. a = 6, b = 8

c. x = 1, y =-1

d. c =4, d = 6

e. x = -2, y =2

f. u = 5, v =5

g. s = 11, t =122

h. x = -2, y =4

Solihull Sixth Form College

Ex 7A

1. $x = 5$
2. $x = 2.5$
3. $x = 1.25$
4. $x = 1.4$
5. $x = 4$
6. $x = 1$
7. $x = 5.25$
8. $x = 1.5$
9. $x = 0.75$
10. $x = 1.5$
11. $x = 4$
12. $x = 1$
13. $x = 2$
14. $x = 8$
15. $x = 1$
16. $x = 7\frac{1}{3}$
17. $x = 5.5$
18. $x = 1$
19. $x = 6.5$
20. $x = 2.5$

Ex 8A

1. $x = 2, x = 0.5$
2. $x = -1, x = 2.5$
3. $x = 5, x = 1.5$
4. $x = -6, x = 5$
5. $x = 1, x = 3$

Ex 8B

1. $3 \pm \sqrt{5}$
2. $\frac{-3\pm\sqrt{13}}{2}$
3. $1 \pm \sqrt{3}$
4. $-4 \pm \sqrt{13}$
5. $\frac{5\pm\sqrt{17}}{2}$
6. $\frac{-3\pm\sqrt{41}}{8}$
7. $\frac{2\pm\sqrt{19}}{3}$
8. $\frac{5\pm\sqrt{41}}{4}$

Ex 8C

Same as Ex 8A

Ex 9A

1. $y = -x + 4$
2. $y = 5x$
3. $y = 4x - 1$
4. $y = -2x + 5$
5. $y = 3x$

Ex 9B

1. $y = \frac{1}{3}x + 4$
2. $y = 2x$
3. $y = 2x + 3$
4. $y = 4x$
5. $y = \frac{1}{2}x + 1$
6. $y = 5 - x$
7. $y = \frac{-3}{2}x + 1$

Ex 9C

1. $par : y = -2x + c \qquad per : y = \frac{1}{2}x + c$
2. $par : y = \frac{-1}{3}x + c \qquad per : y = 3x + c$
3. $par : y = 6x + c \qquad per : y = \frac{-1}{6}x + c$
4. $par : y = \frac{1}{2}x + c \qquad per : y = -2x + c$

Ex 9D

1. (1,6)
2. (4,4)
3. (5,4.5)
4. (5.5,4.5)
5. (1,2)
6. (-5,-7)

Ex 10A

1. x≤7
2. x>2
3. x<4
4. X≤1
5. x≥-1
6. x>0.5
7. x>2.5
8. x≥4
9. -5≥x

10. $2>x$

Ex 10B

1. $X<1$
2. $X>3$
3. $X<-3$
4. $x \geq 4$
5. $x<-18$

Ex 11A

1.$2\sqrt{3}$

2. $10\sqrt{10}$

3. $3\sqrt{5}$

4. $6\sqrt{10}$

5. 2

6. $20\sqrt{10}$

Ex 11B

1. 8
2. $-2\sqrt{3}$
3. $4 + 4\sqrt{3}$
4. $19 + 8\sqrt{3}$
5. $12\sqrt{3} - 9$
6. $8 + 3\sqrt{3}$

Ex 11C

1. $\sqrt{2}\,/2$
2. $\frac{5\sqrt{7}}{7}$
3. $\frac{9\sqrt{5}}{10}$
4. $\frac{3\sqrt{2}}{2}$
5. $\frac{7\sqrt{2}}{10}$
6. $3\sqrt{2}/2$
7. $\frac{4\sqrt{3}}{5}$
8. $2\sqrt{5}$

9. $\frac{5\sqrt{2}}{2}$
10. $\frac{2\sqrt{10}}{15}$

Ex 12A

1. 5cm
2. 11.18cm
3. 5.39cm
4. 11.31cm
5. 10.63cm
6. 13cm

Ex 12B

1. 8cm
2. 5.66cm
3. 16cm
4. 28.91cm
5. 168.93cm
6. 4cm

EX 12C

1. 13cm
2. 5.83cm
3. 6.24cm
4. 24cm
5. 7.64cm
6. 6.34cm

Ex 12D

1. 5.39cm
2. 11.64cm
3. 14.59cm
4. 71.40
5. 42.90

Ex 12E

1. 8.94cm^2
2. 5.20cm^2
3. 19.70cm^2
4. 20.48cm^2
5. 12.06cm^2